dicate that the numerical solutions are more accurate and can be carried farther than either the series expansion solutions or the coordinate expansion solutions.

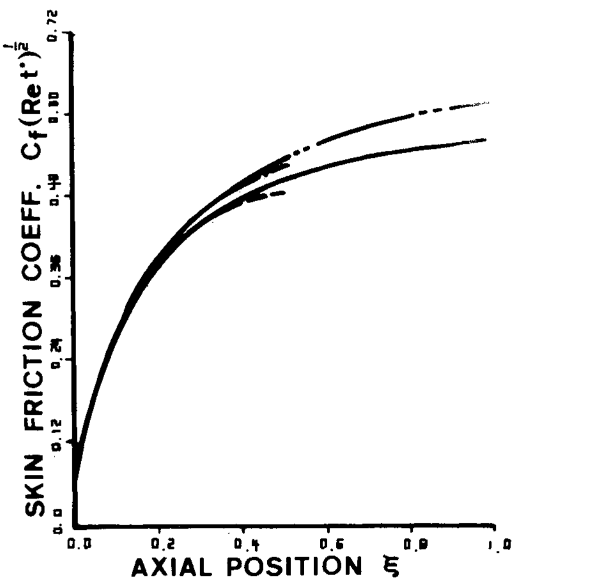

Fig. 2 Skin-friction distribution. Isothermal wall: ----, present solution; ----, Ref. 3. Adiabatic wall: –, present solution; – – –, Ref. 3.

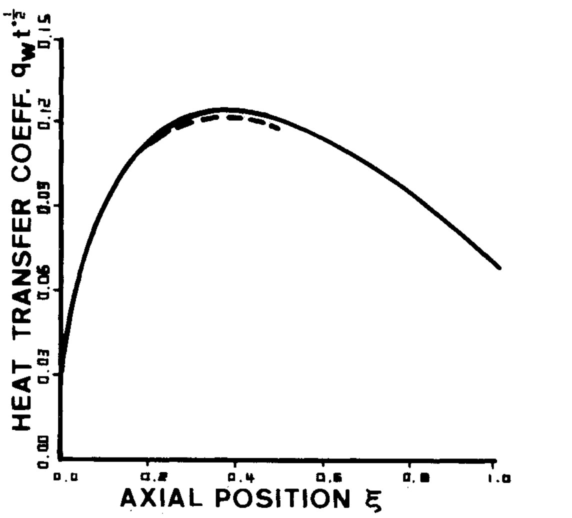

Fig. 3 Heat-transfer distribution for an isothermal wall: –, present solution; – – –, Ref. 3.

Appendix: Inviscid Flow Relation in a Centered Expansion Wave

Velocity:

$$u_\delta^* = \frac{2\xi}{\gamma + 1}$$

Temperature:

$$T_\sigma^* = \left(1\frac{\gamma - 1}{\gamma + 1}\xi\right)^2 \Big/ \gamma$$

Pressure:

$$P_\delta^* = \left(1 - \frac{\gamma - 1}{\gamma + 1}\xi\right)^{[2\gamma/(\gamma - 1)]}$$

Total enthalpy:

$$H_\delta^* = \frac{2}{(3_\gamma - 1)}\left[\left(1 - \frac{\gamma - 1}{\gamma + 1}\xi\right)^2 + \left(\frac{2\xi}{\gamma + 1}\right)\left(\frac{\gamma - 1}{\gamma + 1}\xi\right)\right]$$

References

[1]Hall, J. G., "Laminar Boundary Layers Developed within Unsteady Expansion and Compression Wave," *AIAA Journal*, Vol. 10, April 1972, pp. 499-505.

[2]Chang, L. M. and Chen, C-J., "Unsteady Compressible Laminar Boundary-Layer Flow within a Moving Expansion Wave," *AIAA Journal*, Vol. 19, Dec. 1981, pp. 1551-1557.

[3]Williams, J. C. III and Wang, T. J., "Semi-Similar Solutions of the Unsteady Compressible Laminar Boundary-Layer Equation," *AIAA Journal*, Vol. 23, Feb. 1985, pp. 228-233.

[4]Blottner, F. G., "Finite Difference of Solution of Boundary-Layer Equation," *AIAA Journal*, Vol. 8, Feb. 1970, pp. 193-205.

Extension of Hypersonic, High-Incidence, Slender-Body Similarity

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Introduction

An analysis of inviscid, hypersonic flow past slender bodies at large angles of attack developed by Sychev,[1] shows that these flows are governed by two parameters: the crossflow components of the Mach number and a parameter that relates thickness ratio and angle of attack. The analysis is discussed in well-known texts on hypersonic flow by Hayes and Probstein[2] and Cox and Crabtree.[3]

Recently, Hemsch[4] has shown that the Sychev parameters can be used to correlate experimental normal-force and pitching-moment data for a variety of configurations for Mach numbers from low supersonic to hypersonic. On purpose of this Note is to show that the Sychev analysis is applicable to all slender-body flows with crossflow Mach numbers greater than sonic and hence is not restricted to flows with hypersonic values of the cross flow Mach number as indicated in Refs. 1-3. Also, it will be shown that the Sychev similarity applies to a number of slender-body flows with subsonic crossflow Mach numbers, including incompressible flow.

Near-Field Analysis

The basic Sychev equations governing the inviscid flow in the near field or slender bodies at large angles of attack can be obtained with no assumption regarding Mach number. As shown in Fig. 1, a body-oriented Cartesian coordinate system is used with x in the axial direction and the freestream velocity vector in the x-y plane; the velocity components are u, v, and w. The body thickness and length parameters are d and l, respectively, such that the body thickness ratio δ satisfies the inequality

$$\delta = d/l \ll 1$$

Received Dec. 24, 1986; revision received March 19, 1987.

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Define the nondimensional dependent and independent variables

$$U=\frac{u}{U_\infty \sin\alpha}, \quad V=\frac{v}{U_\infty \sin\alpha}, \quad W=\frac{w}{U_\infty \sin\alpha}$$

$$R=\frac{\rho}{\rho_\infty}, \quad P=\frac{p}{\rho_\infty U_\infty^2 \sin^2\alpha}$$

and

$$X=\frac{x}{l}, \quad Y=\frac{y}{d}, \quad Z=\frac{z}{d}$$

where α is the angle of attack and ρ_∞ and U_∞ the freestream density and speed, respectively. The axial momentum equation can be written as

$$\delta U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+W\frac{\partial U}{\partial Z}+\frac{\delta}{R}\frac{\partial P}{\partial X}=0$$

The simplest form of U that is consistent with this equation, the fact that U has the value $\cot\alpha$ in the freestream, and the assumption that V, W, P, and R are order-one quantities, is

$$U=\cot\alpha+\mathcal{O}(\delta)$$

If the freestream Mach number is supersonic, it is necessary to use the shock relations in obtaining this result. With this form and the neglect of terms of the order δ^2, a set of four equations in the four unknowns V, W, P, and R is obtained as

$$\delta\cot\alpha\frac{\partial R}{\partial X}+V\frac{\partial R}{\partial Y}+W\frac{\partial R}{\partial Z}+R\left(\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}\right)=0$$

$$\delta\cot\alpha\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+W\frac{\partial V}{\partial Z}+\frac{1}{R}\frac{\partial P}{\partial Y}=0$$

$$\delta\cot\alpha\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}+W\frac{\partial W}{\partial Z}+\frac{1}{R}\frac{\partial P}{\partial Z}=0$$

$$\delta\cot\alpha\frac{\partial}{\partial X}\left(\frac{P}{R^\gamma}\right)+V\frac{\partial}{\partial Y}\left(\frac{P}{R^\gamma}\right)+W\frac{\partial}{\partial Z}\left(\frac{P}{R^\gamma}\right)=0$$

where γ is the ratio of specific heats. The boundary condition is

$$\delta\cot\alpha\frac{\partial B}{\partial X}+V_b\frac{\partial B}{\partial Y}+W_b\frac{\partial B}{\partial Z}=0$$

where the subscript b denotes values at the body surface and the body surface is defined by the equation

$$B(X,Y,Z)=0$$

The quantity

$$k_1=\delta\cot\alpha$$

is the Sychev parameter relating the thickness ratio and angle of attack; in general, the value of this parameter is order one for the flowfields of interest here. Since this derivation involves no assumptions concerning the Mach number, these equations govern the near-field solutions of all inviscid slender body flows, regardless of the Mach number.

Strong Shock Wave Applications

As Sychev demonstrated, the shock relations also reduce to a set of four equations in the same four unknowns for shock waves that lie close to the body surface. The procedure of van Dyke[5] can be used to write the shock relations in Cartesian variables. The shock wave surface is defined by the equation

$$S(x,y,z)=0$$

and the velocity component q_n normal to this surface is obtained as

$$q_n=\boldsymbol{q}\cdot\boldsymbol{n}_s=\left(u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}+w\frac{\partial S}{\partial z}\right)\div\sqrt{\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2}$$

where $\boldsymbol{q}$ and $\boldsymbol{n}_s$ are the velocity vector and unit vector normal to the shock, respectively. For shock waves located close to the body, the equations for the conservation of mass, normal momentum, and energy, which all depend on q_n, can be

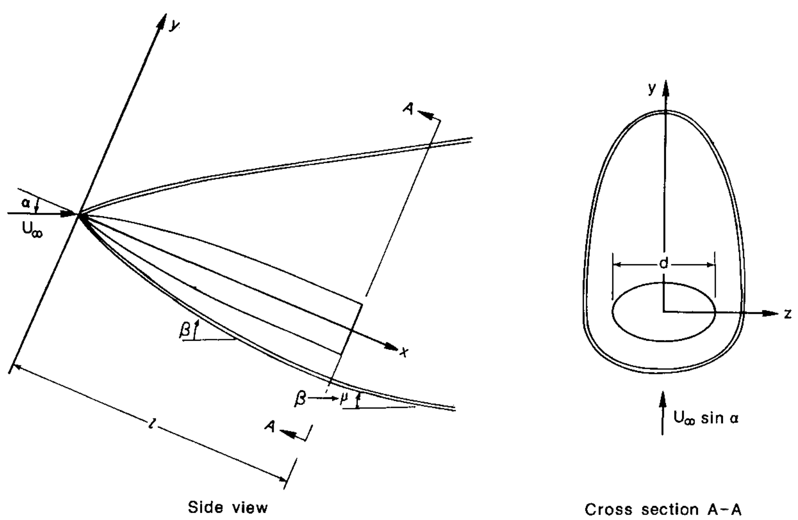

Fig. 1 Geometry and coordinate system.

written to the lowest order in terms of nondimensional, inner-region variables as

$$R\left(\delta \cot\alpha \frac{\partial S}{\partial X} + V\frac{\partial S}{\partial Y} + W\frac{\partial S}{\partial Z}\right) = \delta \cot\alpha \frac{\partial S}{\partial X} + \frac{\partial S}{\partial Y}$$

$$P\left[\left(\frac{\partial S}{\partial Y}\right)^2 + \left(\frac{\partial S}{\partial Z}\right)^2\right] + R\left(\delta \cot\alpha \frac{\partial S}{\partial X} + V\frac{\partial S}{\partial Y} + W\frac{\partial S}{\partial Z}\right)^2$$

$$= \frac{1}{\gamma M_\infty^2 \sin^2\alpha}\left[\left(\frac{\partial S}{\partial Y}\right)^2 + \left(\frac{\partial S}{\partial Z}\right)^2\right]$$

$$+ \left(\delta \cot\alpha \frac{\partial S}{\partial X} + \frac{\partial S}{\partial Y}\right)^2$$

$$\frac{\gamma}{\gamma-1}\frac{P}{R}\left[\left(\frac{\partial S}{\partial Y}\right)^2 + \left(\frac{\partial S}{\partial Z}\right)^2\right] + \frac{1}{2}\left(\delta \cot\alpha \frac{\partial S}{\partial X}\right.$$

$$\left. + V\frac{\partial S}{\partial Y} + W\frac{\partial S}{\partial Z}\right)^2 = \frac{1}{(\gamma-1)M_\infty^2 \sin^2\alpha}\left[\left(\frac{\partial S}{\partial Y}\right)^2\right.$$

$$\left. + \left(\frac{\partial S}{\partial Z}\right)^2\right] + \frac{1}{2}\left(\delta \cot\alpha \frac{\partial S}{\partial X} + \frac{\partial S}{\partial Y}\right)^2$$

With the addition of the tangential shock relation

$$V\frac{\partial S}{\partial Z} - W\frac{\partial S}{\partial Y} = \frac{\partial S}{\partial Z}$$

a set of four equations in four unknowns is obtained. These equations depend only on the ratio of specific heats γ, the Sychev parameter k_1, and the second Sychev parameter

$$k_2 = M_\infty \sin\alpha$$

Thus the solution in the region between a slender body at angle of attack and a closely bounding shock wave can be correlated with the parameters k_1 and k_2.

Sychev[1] concludes that the force and moment coefficients for a slender body with $M_\infty \sin\alpha \gg 1$ can be correlated with k_1 and k_2. For example, the correlation for the normal-force coefficient is

$$C_N = \sin^2\alpha C_N^*(k_1,k_2)$$

He reasons that the shock wave lies close to the body on the windward side, causing the pressure there to be large and dependent on k_1 and k_2. Furthermore, Sychev assumes that the pressure on the leeward side, where the shock wave is far from the body surface (see Fig. 1), is small and almost constant, making no appreciable contribution to the normal force and pitching moment. In fact, flow separation typically causes the pressure on the leeward side of all slender bodies at large angles of attack to be low, regardless of the Mach number.

Hemsch[4] reasons that it is useful to correlate the coefficients so that the gage functions have the proper asymptotic behavior for small values of α. With this philosophy, the normal-force and axial-force coefficients are written as

$$C_N = k_1 \sin^2\alpha \tilde{C}_N(k_1,k_2) \qquad C_A = \delta k_1^2 \sin^2\alpha \tilde{C}_A(k_1,k_2)$$

The reference area for these coefficients is ld. Note that for small values of α, $k_1 \sin^2\alpha$ varies as α and $k_1^2 \sin^2\alpha$ is constant.

The Sychev similarity is valid for any supersonic, slender-body flow for which the angle of attack exceeds the freestream Mach angle, which is defined as

$$\mu = \sin^{-1}(1/M_\infty)$$

As illustrated in the figure, the shock wave will be close to the body on the windward side if α and M_∞ are related as

$$\sin\alpha > \sin\beta_{\min} = \sin\mu = 1/M_\infty$$

or

$$M_\infty \sin\alpha > 1$$

where $\beta_{\min}$ is the minimum value of the shock angle. This inequality defines a Mach number range that is expanded enormously over that permitted by the hypersonic limit $M_\infty \sin\alpha \gg 1$ given by Sychev.

The Sychev formulation is applicable for subsonic values of the crossflow Mach number ($M_\infty \sin\alpha < 1$) if the flow is conventionally hypersonic [$M_\infty \gg 1$, $M_\infty\delta \geqq \mathcal{O}(1)$]. For these flows $\alpha \leqq \mathcal{O}(\delta)$, so that δ characterizes all slopes, and thus d is the proper transverse length scale for the whole shock region.[2,3] Therefore, the approximate forms for the governing equations and shock relations developed by Sychev pertain.

Potential Flow Applications

A related similarity formulation involving the Sychev parameter k_1 can be established for several potential flows. It has already been shown that the flow in the region near the body is governed by the Sychev equations and thus depends on the parameter k_1. It remains to determine the nature of the flow far from the body.

The potential flow solution in the outer region of a slender body at angle of attack does not depend simply on the Sychev parameters k_1 and k_2. Far from the body, the length scales of all cordinates can be taken as l for potential flow, so nondimensional coordinates of the form

$$\xi = \frac{x}{l} \qquad \eta = \frac{y}{l} \qquad \zeta = \frac{z}{l}$$

can be assumed. A perturbation velocity potential is defined such that

$$U = \cot\alpha + \delta\frac{\partial\phi}{\partial\xi} \qquad V = 1 + \delta\frac{\partial\phi}{\partial\eta} \qquad W = \delta\frac{\partial\phi}{\partial\zeta}$$

This form is consistent with the established form of U and the equations for zero vorticity. The linearized form of the potential equation is

$$(1 - M_\infty^2 \cos^2\alpha)\frac{\partial^2\phi}{\partial\xi^2} - 2M_\infty^2 \sin\alpha \cos\alpha \frac{\partial^2\phi}{\partial\xi\partial\eta}$$

$$+ (1 - M_\infty^2 \sin^2\alpha)\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\zeta^2} = 0$$

The coefficients of this equation do not depend on k_1 at all and cannot be expressed simply as functions of $k_2 = M_\infty \sin\alpha$.

A reduced number of parameters govern the solutions for two types of potential flow. For incompressible flow, the outer region is governed by the Laplace equation, which is parameterless. For subsonic and supersonic flows that satisfy the inequality $M_\infty \sin\alpha \ll 1$, the governing equation in the outer region is

$$(1 - M_\infty^2)\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\zeta^2} = 0$$

so the solution depends on the Mach number. As stated previously, the inner solution depends on the Sychev parameter $k_1 = \delta \cot\alpha$. For both types of flow discussed, the governing equations in the inner region can be written in terms of the perturbation velocity potential as

$$\frac{\partial^2\phi}{\partial Y^2} + \frac{\partial^2\phi}{\partial Z^2} = 0$$

$$P - P_\infty = -\delta \cot\alpha \frac{\partial\phi}{\partial X} - \frac{\partial\phi}{\partial Y} - \frac{1}{2}\left(\frac{\partial\phi}{\partial Y}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial Z}\right)^2$$

where

$$P_\infty = 1/\gamma M_\infty^2 \sin^2\alpha$$

Note that these equations can also be obtained with slender-wing and slender-body theory.[6] It is concluded that the normal-force and axial-force coefficients for incompressible flow and for subsonic and supersonic flows with small values of the crossflow Mach number can be expressed as

$$C_N = k_1 \sin^2\alpha \hat{C}_N(k_1, M_\infty) \qquad C_A = \delta k_1^2 \sin^2\alpha \hat{C}_A(k_1, M_\infty)$$

As slender-wing and slender-body theory shows, the dependence of C_N on M_∞ is weak and can be neglected for potential flow. However, the dependence of C_A on M_∞ is not negligible.

References

[1]Sychev, V. V., "Three-Dimensional Hypersonic Gas Flow Past Slender Bodies at High Angles of Attack," *Prikladnaia Matematika i Mekhanika*, Vol. 24, 1960, pp. 205–212.

[2]Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory, Vol. I: Inviscid Flows*, Academic Press, New York, 1966.

[3]Cox, R. N. and Crabtree, L. F., *Elements of Hypersonic Aerodynamics*, Academic Press, New York, 1965.

[4]Hemsch, M. J., "Engineering Analysis of Slender Body Aerodynamics Using Sychev Similarity Parameters," AIAA Paper 87-0267, Jan. 1987.

[5]Van Dyke, M. D., "A Study of Hypersonic Small Disturbance Theory," NACA TR 1194, 1954.

[6]Ashley, H. and Landahl, M., *Aerodynamics of Wings and Bodies,* Addison-Wesley, Inc., Reading, MA, 1965.

Similarity Rule for Sidewall Boundary-Layer Effects in Airfoil Testing

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Introduction

THE sidewall boundary-layer interference in testing of airfoils in wind tunnels has recently been the subject of considerable attention. Earlier methods to account for the sidewall effects were based on the vorticity model proposed by Preston.[1] However, following recent experimental observations made in the ONERA[2] tunnel, Barnwell,[3,4] and Winter and Smith[5] have independently proposed theories based on the changes in the sidewall boundary-layer thickness due to the airfoil flowfield. In the form proposed by Barnwell, a factor similar to the Prandtl-Glauert rule was suggested to account for the sidewall boundary-layer effects. This was later extended to transonic speeds by Sewall[6] by using the von Kármán similarity rule. In this Note, an alternative simpler form of the similarity rule is presented by considering the sidewall boundary layer to cause changes in both the airfoil thickness and the freestream Mach number. This approach, within the small-disturbance approximation, encompasses both the methods of Barnwall and Sewall and, hence, can be used from low speeds to transonic speeds.

Received Feb. 6, 1987; revision received April 6, 1987.

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Analysis

For the flow over an airfoil mounted between the walls of a two-dimensional wind tunnel of width b, the sidewall boundary-layer effects can be represented in a simplified manner by the small-disturbance equation[4,6,7]

$$(1 - M^2 + k)\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0 \tag{1}$$

where x, y refer to the streamwise and normal coordinates, u, v are the perturbation velocities, and M is the local Mach number. In arriving at Eq. (1), it was assumed that the equivalent flat-plate Reynolds number for the sidewall boundary layer was much larger than the airfoil chord Reynolds number, and the changes in the boundary-layer thickness introduced cross-flow velocities that varied linearly across the tunnel width. The parameter k is nearly constant and is given by the values of the undisturbed sidewall boundary-layer displacement thickness (δ^*) and the shape factor H.

$$k = (2\delta^*/b)(2 + 1/H - M^2) \tag{2}$$

Introducing the coordinate transformation $\xi = x$ and $\eta = y(1+k)^{1/2}$, and the velocity potential ϕ, Eq. (1) can be reduced to an equivalent two-dimensional flow represented by

$$(1 - M_e^2)\phi_{\xi\xi} + \phi_{\eta\eta} = 0 \tag{3}$$

where $M_e = M/(1+k)^{1/2}$ is the local Mach number in the equivalent flow. If the freestream velocity is U_∞ and the airfoil thickness is τ, the corresponding boundary condition on the airfoil surface is given by

$$(\phi_\eta)_{\eta=0} = U_\infty \tau (1+k)^{-1/2} f'(x/c) \tag{4}$$

where $f(x/c)$ represents the airfoil shape. From the transformed boundary condition (4), it follows that, in the equivalent flow represented by Eq. (3), either the freestream velocity or the airfoil thickness ratio can be considered to be reduced by a factor of $(1+k)^{-1/2}$. For subsonic flow, Eq. (3) can be linearized by approximating M by the freestream value M everywhere. The corresponding freestream Mach number M_c in the equivalent two-dimensional flow will be $M_c = M_\infty/(1+k)^{1/2}$.

In transonic flow, the Mach number is reduced everywhere by the factor $(1+k)^{-1/2}$. However, for both subsonic and transonic flows, if the freestream velocity in the equivalent flow is U, it follows from Eq. (4) that the equivalent flow corresponds to flow over a thinner profile with a thickness ratio of $\tau_e = \tau/(1+k)^{1/2}$. This equivalent two-dimensional flow can be related to a number of other two-dimensional flows by using the high-speed similarity rules, and the results of Refs. 3 and 6 can be obtained as particular cases.[7] For subsonic flow, if the freestream velocity in the equivalent flow is U_e $[= U_\infty/(1+k)^{1/2}]$, the corresponding pressure